Simulation of fault-tolerant quantum circuits on quantum computational tensor network

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In the framework of quantum computational tensor network [D. Gross and J. Eisert, Phys. Rev. Lett. **98**, 220503 (2007)], which is a general framework of measurement-based quantum computation, the resource many-body state is represented in a tensor-network form (or a matrix-product form), and universal quantum computation is performed in a virtual linear space, which is called a correlation space, where tensors live. Since any unitary operation, state preparation, and the projection measurement in the computational basis can be simulated in a correlation space, it is natural to expect that fault-tolerant quantum circuits can also be simulated in a correlation space. However, we point out that not all physical errors on physical qudits appear as linear completely-positive trace-preserving errors in a correlation space. Since the theories of fault-tolerant quantum circuits known so far assume such noises, this means that the simulation of fault-tolerant quantum circuits in a correlation space is not so straightforward for general resource states.

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I. INTRODUCTION

Quantum many-body states, which have long been central research objects in condensed matter physics, statistical physics, and quantum chemistry, are now attracting the renewed interest in quantum information science as fundamental resources for quantum information processing. One of the most celebrated examples is one-way quantum computation [1–3]. Once the highly-entangled many-body state which is called the cluster state is prepared, universal quantum computation is possible with adaptive local measurements on each qubit. Recently, the concept of quantum computational tensor network (QCTN) [4–6], which is the general framework of measurement-based quantum computation on quantum many-body states, was proposed. This novel framework has enabled us to understand how general measurement-based quantum computation is performed on many other resource states beyond the cluster state. The most innovative feature of QCTN is that the resource state is represented in a tensor-network form (or a matrix-product form) [7–9], and universal quantum computation is performed in the virtual linear space where tensors live. For example, let us consider the one-dimensional open-boundary chain of N qudits in the matrix-product form

$$|\Psi(L,R)\rangle_1^N \equiv \frac{1}{\sqrt{f_N(|L\rangle,|R\rangle)}} \sum_{k_1=0}^{d-1} ... \sum_{k_N=0}^{d-1} \langle L|A[k_N]...A[k_1]|R\rangle |k_N,...,k_1\rangle, \tag{1}$$

where

$$f_N(|L\rangle,|R\rangle) \equiv \langle L|(\mathcal{A}^N|R\rangle\langle R|)|L\rangle$$

is the normalization factor,

$$\mathcal{A}\rho \equiv \sum_{i=0}^{d-1} A[i]\rho A^\dagger[i]$$

is a map, $\{|0\rangle, ..., |d-1\rangle\}$ is a certain basis in the d-dimensional Hilbert space ($2 \leq d < \infty$), $|L\rangle$ and $|R\rangle$ are D-dimensional complex vectors, and $\{A[0], ..., A[d-1]\}$ are $D \times D$ complex matrices. Let us also define the projection measurement $\mathcal{M}_{\theta,\phi}$ on a single physical qudit by

$$\mathcal{M}_{\theta,\phi} \equiv \left\{ |\alpha_{\theta,\phi}\rangle, |\beta_{\theta,\phi}\rangle, |2\rangle, ..., |d-1\rangle \right\}, \tag{2}$$

where

$$|\alpha_{\theta,\phi}\rangle \equiv \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle,$$

$$|\beta_{\theta,\phi}\rangle \equiv \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle,$$

$0 < \theta < \pi$, and $0 \le \phi < 2\pi$. If we do the measurement $\mathcal{M}_{\theta,\phi}$ on the first physical qudit of $|\Psi(L,R)\rangle_1^N$ and if the first physical qudit is projected onto, for example, $|\alpha_{\theta,\phi}\rangle$ as a result of this measurement, the state $|\Psi(L,R)\rangle_1^N$ becomes

$$\frac{1}{\sqrt{f_{N-1}(|L\rangle, A[\alpha_{\theta,\phi}]|R\rangle)}} \sum_{k_2=0}^{d-1} ... \sum_{k_N=0}^{d-1} \langle L|A[k_N]...A[k_2]A[\alpha_{\theta,\phi}]|R\rangle |k_N,...,k_2\rangle \otimes |\alpha_{\theta,\phi}\rangle$$

$$= \frac{1}{\sqrt{f_{N-1}(|L\rangle, A[\alpha_{\theta,\phi}]|R\rangle)}} \|A[\alpha_{\theta,\phi}]\| \sum_{k_2=0}^{d-1} ... \sum_{k_N=0}^{d-1} \langle L|A[k_N]...A[k_2]\frac{A[\alpha_{\theta,\phi}]}{\|A[\alpha_{\theta,\phi}]\|}|R\rangle |k_N,...,k_2\rangle \otimes |\alpha_{\theta,\phi}\rangle,$$

where

$$A[\alpha_{\theta,\phi}] \equiv \cos\frac{\theta}{2} A[0] + e^{-i\phi} \sin\frac{\theta}{2} A[1].$$

Then, we say "the operation

$$|R\rangle \to \frac{A[\alpha_{\theta,\phi}]}{\|A[\alpha_{\theta,\phi}]\|}|R\rangle$$

is implemented in the correlation space". In particular, if $A[0]$, $A[1]$, θ, and ϕ are appropriately chosen in such a way that $A[\alpha_{\theta,\phi}]$ is proportional to a unitary, we can "simulate" the unitary evolution

$$\frac{A[\alpha_{\theta,\phi}]}{\|A[\alpha_{\theta,\phi}]\|}|R\rangle$$

of the vector $|R\rangle$ in the virtual linear space where A's, $|R\rangle$, and $|L\rangle$ live. This virtual linear space is called the correlation space [4–6, 10–12]. The core of QCTN is this "virtual quantum computation" in the correlation space. If the correlation space has a sufficient structure and if A's, $|L\rangle$, and $|R\rangle$ are appropriately chosen, we can "simulate" universal quantum circuit in the correlation space [4–6, 10–12].

For the realization of a scalable quantum computer, a theory of fault-tolerant (FT) quantum computation [13–17] is necessary. In fact, several researches have been performed on FT quantum computation in the one-way model [3, 18–22]. However, there has been no result about a theory of FT quantum computation on general QCTN [23]. In particular, there is severe lack of knowledge about FT quantum computation on resource states with $d \ge 3$. It is necessary to consider resource states with $d \ge 3$ if we want to enjoy the cooling preparation of a resource state and the energy-gap protection of measurement-based quantum computation with a physically natural Hamiltonian, since no genuinely entangled qubit state can be the unique ground state of a two-body frustration-free Hamiltonian [28].

One straightforward way of implementing FT quantum computation on QCTN is to encode physical qudits with a quantum error correcting code:

$$|\tilde{\Psi}\rangle \equiv \frac{1}{\sqrt{f_N(L,R)}} \sum_{k_1=0}^{d-1} ... \sum_{k_N=0}^{d-1} \langle L|A[k_N]...A[k_1]|R\rangle |\tilde{k}_N,...,\tilde{k}_1\rangle,$$

where $|\tilde{k}_i\rangle$ $(i = 1,...,N)$ is the encoded version of $|k_i\rangle$ (such as $|\tilde{0}\rangle = |000\rangle$ and $|\tilde{1}\rangle = |111\rangle$, etc.) In fact, this strategy was taken in Refs. [21, 22] for the one-way model $(d = 2)$, and it was shown there that a FT construction of the encoded cluster state is possible. For $d \ge 3$, however, such a strategy is difficult, since theories of quantum error correcting codes and FT preparations of the encoded resource state $|\tilde{\Psi}\rangle$ are less developed for $d \ge 3$. Furthermore, if we encode physical qudits with a quantum error correcting code, the parent Hamiltonian should no longer be two-body interacting one.

The other way of implementing FT quantum computation on QCTN is to simulate FT quantum circuits in the correlation space. Since any unitary operation, state preparation, and the projective measurement in the computational basis can be simulated in a correlation space (for a more precise discussion about the possibility of the measurement, see Ref. [12]), it is natural to expect that FT quantum circuits can also be simulated in a correlation space. An advantage of this strategy is that theories of FT quantum circuits for qubit systems are well developed [13–17]. In fact, this strategy was taken in Refs. [18, 19] for the one-way model $(d = 2)$. They introduced a method (which we call "the ensemble method" since the ensemble of all measurement results are considered) of simulating quantum circuits in the correlation space of the cluster state, and showed that all physical errors on physical qubits can be linear completely-positive trace-preserving (CPTP) maps in the correlation space of the cluster state. This means that FT quantum circuits can be simulated in the correlation space of the cluster state.

In this paper, however, we point out that it is not so straightforward to simulate FT quantum circuits in a correlation space of a general resource state. In the next section, Sec. II, we review the simulation of FT quantum circuits on the one-dimensional cluster state [18, 19] in terms of the QCTN picture to fix the notation. We see that for the cluster state all physical errors can be linear completely-positive trace-preserving (CPTP) maps in the correlation space, and therefore the theory of FT quantum circuits can be used in the correlation space. However, this is not the case for other general resource states of QCTN. As an example, we consider a similar way of simulating quantum circuits in the correlation space of the one-dimensional AKLT state [29, 30] in Sec. III, and show that not all physical errors can be linear CPTP maps in the correlation space of the AKLT state. Since all theories of FT quantum circuits known so far assume such noises [13–17], this means that it is not so straightforward to apply these FT theories to quantum circuits simulated in the correlation space of general QCTN. In Sec. IV, we give some intuitive explanations of the reason why the cluster state is so special, and why not all resource states work as the cluster state. In Sec. V, we consider another standard way of simulating quantum circuits in the correlation space, which we call "the trajectory method" since a specific trajectory (measurement results) is considered. However, we show a general theorem that such an another way does neither work if $d \geq 3$.

In short, we show in this paper that it is not so straightforward to simulate FT quantum circuits in the correlation space of a general resource state. Since all errors behave nicely in the correlation space of the cluster state [18, 19], less attention has been paid to the difference between a real physical space and a correlation space of a general resource state. Our results here suggest that these two spaces can be different, and because of the difference, simulations of FT quantum circuits can be difficult in a correlation space. Of course, we do not show here the impossibility of making a QCTN fault-tolerant. In a future, a highly elaborated method might be found which makes all QCTN fault-tolerant. We hope that our results will help to study such a challenging subject of a future study.

Assumptions: Throughout this paper, we make the following assumptions: Since the MPS $|\Psi(L, R)\rangle_1^N$, Eq. (1), is a resource state for measurement-based quantum computation, we can assume without loss of generality that $A[\alpha_{\theta,\phi}]$, $A[\beta_{\theta,\phi}]$, $A[2]$, $A[3]$, ..., $A[d-1]$ are unitary up to constants:

$$
\begin{aligned}
A[\alpha_{\theta,\phi}] &= c_\alpha U_\alpha, \\
A[\beta_{\theta,\phi}] &= c_\beta U_\beta, \\
A[2] &= c_2 U_2, \\
A[3] &= c_3 U_3, \\
&\quad ... \\
A[d-1] &= c_{d-1} U_{d-1},
\end{aligned}
\tag{3}
$$

where c_α, c_β, c_2, ... c_{d-1} are real positive numbers, U_α, U_β, U_2, ..., U_{d-1} are unitary operators, and

$$
A[\beta_{\theta,\phi}] \equiv \sin\frac{\theta}{2} A[0] - e^{-i\phi}\cos\frac{\theta}{2} A[1].
$$

This means that any operation implemented in the correlation space by the measurement $\mathcal{M}_{\theta,\phi}$ on a single physical qudit of $|\Psi(L, R)\rangle_1^N$ is unitary. Note that this assumption is reasonable, since otherwise $|\Psi(L, R)\rangle_1^N$ does not seem to be useful as a resource for measurement-based quantum computation. In fact, all known resource states so far [1–6, 29, 31–33], including the cluster state and the AKLT state, satisfy this assumption by appropriately rotating each local physical basis. Furthermore, we can take c_α, c_β, c_2,..., c_{d-1} such that

$$
C \equiv c_\alpha^2 + c_\beta^2 + \sum_{k=2}^{d-1} c_k^2 = 1,
$$

since

$$
\frac{1}{\sqrt{f_N(|L\rangle, |R\rangle)}} \sum_{k_1,...,k_N} \langle L|A[k_N]...A[k_1]|R\rangle |k_N, ..., k_1\rangle = \frac{1}{\sqrt{f_N(|L\rangle, |R\rangle)}} \sqrt{C}^N \sum_{k_1,...,k_N} \langle L|\frac{A[k_N]}{\sqrt{C}}...\frac{A[k_1]}{\sqrt{C}}|R\rangle |k_N, ..., k_1\rangle
$$

and we can redefine $A[k_i]/\sqrt{C} \to A[k_i]$.

II.   SIMULATION ON THE CLUSTER STATE

Let us first review the results for the cluster state [18, 19] in terms of the QCTN picture to fix the notation.

A. Simulation on the cluster state without error

Let us first assume that there is no error. The one-dimensional cluster state is the matrix-product state defined by $d = 2$, $A[0] = |+\rangle\langle 0|$, and $A[1] = |-\rangle\langle 1|$. We measure each physical qubit in the basis

$$|\theta_s\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + (-1)^s e^{i\theta}|1\rangle),$$

where $s \in \{0, 1\}$. In the correlation space, $X^s J(\theta)$, where $J(\theta) \equiv He^{i\theta Z/2}$, is implemented. First, we measure the first physical qubit in the $\{|\theta_0\rangle, |\theta_1\rangle\}$ basis. Then we obtain

$$\frac{1}{2f_N(|L\rangle, |R\rangle)} \sum_{s_1=0}^{1} W(X^{s_1} J(\theta)|R\rangle)_2 \otimes |\theta_{s_1}\rangle\langle\theta_{s_1}| \otimes m(s_1),$$

where $m(0)$ and $m(1)$ are mutually orthogonal states which record the measurement result, and

$$W(|\psi\rangle)_r \equiv \sum_{k_r,...,k_N} \sum_{k'_r,...,k'_N} \langle L|A[k_N]...A[k_r]|\psi\rangle\langle\psi|A^\dagger[k'_r]...A^\dagger[k'_N]|L\rangle|k_N, ..., k_r\rangle\langle k'_N, ..., k'_r|.$$

If we trace out the measured first physical qubit $|\theta_{s_1}\rangle\langle\theta_{s_1}|$, we obtain

$$\frac{1}{2f_N(|L\rangle, |R\rangle)} \sum_{s_1=0}^{1} W(X^{s_1} J(\theta)|R\rangle)_2 \otimes m(s_1).$$

Second, we measure the second physical qubit in the $\{X^{s_1}|\phi_0\rangle, X^{s_1}|\phi_1\rangle\}$ basis. Then we obtain

$$\frac{1}{2^2 f_N(|L\rangle, |R\rangle)} \sum_{s_2=0}^{1} \sum_{s_1=0}^{1} W(X^{s_2} J((-1)^{s_1}\phi) X^{s_1} J(\theta)|R\rangle)_3 \otimes X^{s_1}|\phi_{s_2}\rangle\langle\phi_{s_2}|X^{s_1} \otimes m(s_1) \otimes m(s_2)$$

$$= \frac{1}{2^2 f_N(|L\rangle, |R\rangle)} \sum_{s_2=0}^{1} \sum_{s_1=0}^{1} W(X^{s_2} Z^{s_1} J(\phi) J(\theta)|R\rangle)_3 \otimes X^{s_1}|\phi_{s_2}\rangle\langle\phi_{s_2}|X^{s_1} \otimes m(s_1) \otimes m(s_2).$$

If we trace out the measured second physical qubit $X^{s_1}|\phi_{s_2}\rangle\langle\phi_{s_2}|X^{s_1}$, we obtain

$$\frac{1}{2^2 f_N(|L\rangle, |R\rangle)} \sum_{s_2=0}^{1} \sum_{s_1=0}^{1} W(X^{s_2} Z^{s_1} J(\phi) J(\theta)|R\rangle)_3 \otimes m(s_1) \otimes m(s_2).$$

Third, we measure the third physical qubit in the $\{Z^{s_1} X^{s_2}|\eta_0\rangle, Z^{s_1} X^{s_2}|\eta_1\rangle\}$ basis. Then we obtain

$$\frac{1}{2^3 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{1} \sum_{s_2=0}^{1} \sum_{s_1=0}^{1} W(X^{s_3} X^{s_1} J((-1)^{s_2}\eta) X^{s_2} Z^{s_1} J(\phi) J(\theta)|R\rangle)_4 \otimes Z^{s_1} X^{s_2}|\eta_{s_3}\rangle\langle\eta_{s_3}|X^{s_2} Z^{s_1}$$

$$\otimes m(s_1) \otimes m(s_2) \otimes m(s_3)$$

$$= \frac{1}{2^3 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{1} \sum_{s_2=0}^{1} \sum_{s_1=0}^{1} W(X^{s_3} Z^{s_2} J(\eta) J(\phi) J(\theta)|R\rangle)_4 \otimes Z^{s_1} X^{s_2}|\eta_{s_3}\rangle\langle\eta_{s_3}|X^{s_2} Z^{s_1}$$

$$\otimes m(s_1) \otimes m(s_2) \otimes m(s_3).$$

If we trace out the measured third physical qubit $Z^{s_1} X^{s_2}|\eta_{s_3}\rangle\langle\eta_{s_3}|X^{s_2} Z^{s_1}$, we obtain

$$\frac{1}{2^3 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{1} \sum_{s_2=0}^{1} \sum_{s_1=0}^{1} W(X^{s_3} Z^{s_2} J(\eta) J(\phi) J(\theta)|R\rangle)_4 \otimes m(s_1) \otimes m(s_2) \otimes m(s_3)$$

$$= \frac{1}{2^3 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{1} \sum_{s_2=0}^{1} W(X^{s_3} Z^{s_2} J(\eta) J(\phi) J(\theta)|R\rangle)_4 \otimes I \otimes m(s_2) \otimes m(s_3).$$

In this way, we can simulate the desired unitary operation $J(\eta) J(\phi) J(\theta)$ on $|R\rangle$ up to Pauli byproducts $X^{s_3} Z^{s_2}$ in the correlation space. These Pauli byproducts can be corrected later, since they are specified by $m(s_2) \otimes m(s_3)$.

B.   Effect of a CPTP error on a physical qudit of a general state

Before studying the simulation of quantum circuits on the cluster state with error, let us consider the effect of a CPTP error on a physical qudit of general resource states, since we will use it later. Let us assume that a CPTP error

$$\rho \to \sum_{j=1}^{w} F_j \rho F_j^\dagger,$$

where $\sum_{j=1}^{w} F_j^\dagger F_j = I$, occurs on the first physical qudit of $|\Psi(L,R)\rangle_1^N$:

$$\frac{1}{f_N(|L\rangle,|R\rangle)} \sum_j \sum_{k_1,...,k_N} \sum_{k_1',...,k_N'} \langle L|A[k_N]...A[k_1]|R\rangle\langle R|A^\dagger[k_1']...A^\dagger[k_N']|L\rangle|k_N,...,k_2\rangle\langle k_2',...,k_N'| \otimes F_j|k_1\rangle\langle k_1'|F_j^\dagger.$$

If we measure the first physical qudit in a certain basis $\{|m_s\rangle\}$,

$$\frac{1}{f_N(|L\rangle,|R\rangle)} \sum_{j,s} \sum_{k_1,...,k_N} \sum_{k_1',...,k_N'} \langle L|A[k_N]...A[k_1]|R\rangle\langle R|A^\dagger[k_1']...A^\dagger[k_N']|L\rangle|k_N,...,k_2\rangle\langle k_2',...,k_N'|$$

$$\otimes|m_s\rangle\langle m_s|F_j|k_1\rangle\langle k_1'|F_j^\dagger|m_s\rangle\langle m_s|$$

$$= \frac{1}{f_N(|L\rangle,|R\rangle)} \sum_{j,s} \sum_{k_2,...,k_N} \sum_{k_2',...,k_N'} \langle L|A[k_N]...A[k_2]E_{j,s}|R\rangle\langle R|E_{j,s}^\dagger A^\dagger[k_2']...A^\dagger[k_N']|L\rangle|k_N,...,k_2\rangle\langle k_2',...,k_N'| \otimes |m_s\rangle\langle m_s|$$

$$= \frac{1}{f_N(|L\rangle,|R\rangle)} \sum_s \left(\sum_j W(E_{j,s}|R\rangle)_2 \right) \otimes |m_s\rangle\langle m_s|,$$

where

$$E_{j,s} \equiv \sum_k A[k]\langle m_s|F_j|k\rangle.$$

If we trace out $|m_s\rangle$, we obtain

$$\frac{1}{f_N(|L\rangle,|R\rangle)} \sum_s \sum_j W(E_{j,s}|R\rangle)_2$$

$$= \frac{1}{f_N(|L\rangle,|R\rangle)} \sum_{k_2,...,k_N} \sum_{k_2',...,k_N'} \langle L|A[k_N]...A[k_2]\left(\sum_{j,s} E_{j,s}|R\rangle\langle R|E_{j,s}^\dagger\right) A^\dagger[k_2']...A^\dagger[k_N']|L\rangle|k_N,...,k_2\rangle\langle k_2',...,k_N'|.$$

This means that the map

$$|R\rangle\langle R| \to \sum_{s,j} E_{j,s}|R\rangle\langle R|E_{j,s}^\dagger \qquad (4)$$

is implemented in the correlation space. Note that

$$\sum_{j,s} E_{j,s}^\dagger E_{j,s} = \sum_{j,s,k,k'} A^\dagger[k]A[k']\langle k|F_j^\dagger|m_s\rangle\langle m_s|F_j|k'\rangle$$

$$= \sum_{k,k'} A^\dagger[k]A[k']\langle k|k'\rangle$$

$$= \sum_{k,k'} A^\dagger[k]A[k']\langle k|U_{\mathcal{M}_{\theta,\phi}} U_{\mathcal{M}_{\theta,\phi}}^\dagger|k'\rangle$$

$$= A^\dagger[\alpha_{\theta,\phi}]A[\alpha_{\theta,\phi}] + A^\dagger[\beta_{\theta,\phi}]A[\beta_{\theta,\phi}] + \sum_{k=2}^{d-1} A^\dagger[k]A[k]$$

$$= I,$$

where

$$U_{\mathcal{M}_{\theta,\phi}} \equiv |\alpha_{\theta,\phi}\rangle\langle 0| + |\beta_{\theta,\phi}\rangle\langle 1| + \sum_{k=2}^{d-1} |k\rangle\langle k|$$

is a unitary operator. Therefore, the map Eq. (4) is CPTP.

Note that this result does not mean that we can always have CPTP errors in the correlation space: In this section, we did not consider any quantum gate. If we implement quantum gates in the correlation space, the situation becomes more complicated, and, as we will see later, we sometimes have non-CPTP errors in the correlation space.

C.   Simulation on the cluster state with error

Now let us consider the case where we implement quantum gates on the cluster state with error. We assume that a CPTP error occurs on the first physical qubit of the one-dimensional cluster state. If we measure the first physical qubit in the $\{|\theta_0\rangle, |\theta_1\rangle\}$ basis after such an error, we obtain

$$\frac{1}{f_N(|L\rangle,|R\rangle)} \sum_{s_1=0}^{1} \Big(\sum_j W(E_{j,s_1}|R\rangle)_2 \Big) \otimes |\theta_{s_1}\rangle\langle\theta_{s_1}| \otimes m(s_1).$$

By tracing out the first physical qubit $|\theta_{s_1}\rangle\langle\theta_{s_1}|$, we obtain

$$\frac{1}{f_N(|L\rangle,|R\rangle)} \sum_{s_1=0}^{1} \Big(\sum_j W(E_{j,s_1}|R\rangle)_2 \Big) \otimes m(s_1).$$

Second, we measure the second physical qubit in the $\{X^{s_1}|\phi_0\rangle, X^{s_1}|\phi_1\rangle\}$ basis. Then we obtain

$$\frac{1}{2f_N(|L\rangle,|R\rangle)} \sum_{s_2=0}^{1}\sum_{s_1=0}^{1} \Big(\sum_j W(X^{s_2}J((-1)^{s_1}\phi)E_{j,s_1}|R\rangle)_3 \Big) \otimes X^{s_1}|\phi_{s_2}\rangle\langle\phi_{s_2}|X^{s_1} \otimes m(s_1) \otimes m(s_2).$$

By tracing out the second physical qubit $X^{s_1}|\phi_{s_2}\rangle\langle\phi_{s_2}|X^{s_1}$,

$$\frac{1}{2f_N(|L\rangle,|R\rangle)} \sum_{s_2=0}^{1}\sum_{s_1=0}^{1} \Big(\sum_j W(X^{s_2}J((-1)^{s_1}\phi)E_{j,s_1}|R\rangle)_3 \Big) \otimes m(s_1) \otimes m(s_2).$$

Third, we measure the third physical qubit in the $\{Z^{s_1}X^{s_2}|\eta_0\rangle, Z^{s_1}X^{s_2}|\eta_1\rangle\}$ basis. Then we obtain

$$\frac{1}{2^2 f_N(|L\rangle,|R\rangle)} \sum_{s_3,s_2,s_1} \Big(\sum_j W(X^{s_3}X^{s_1}J((-1)^{s_2}\eta)X^{s_2}J((-1)^{s_1}\phi)E_{j,s_1}|R\rangle)_4 \Big) \otimes Z^{s_1}X^{s_2}|\eta_{s_3}\rangle\langle\eta_{s_3}|X^{s_2}Z^{s_1}$$
$$\otimes m(s_1) \otimes m(s_2) \otimes m(s_3)$$
$$= \frac{1}{2^2 f_N(|L\rangle,|R\rangle)} \sum_{s_3,s_2,s_1} \Big(\sum_j W(X^{s_3}X^{s_1}Z^{s_2}J(\eta)J((-1)^{s_1}\phi)E_{j,s_1}|R\rangle)_4 \Big) \otimes Z^{s_1}X^{s_2}|\eta_{s_3}\rangle\langle\eta_{s_3}|X^{s_2}Z^{s_1}$$
$$\otimes m(s_1) \otimes m(s_2) \otimes m(s_3).$$

If we trace out the third physical qubit $Z^{s_1}X^{s_2}|\eta_{s_3}\rangle\langle\eta_{s_3}|X^{s_2}Z^{s_1}$,

$$\frac{1}{2^2 f_N(|L\rangle,|R\rangle)} \sum_{s_3,s_2,s_1} \Big(\sum_j W(X^{s_3}X^{s_1}Z^{s_2}J(\eta)J((-1)^{s_1}\phi)E_{j,s_1}|R\rangle)_4 \Big) \otimes m(s_1) \otimes m(s_2) \otimes m(s_3).$$

If we further trace out the first record state $m(s_1)$,

$$\frac{1}{2^2 f_N(|L\rangle,|R\rangle)} \sum_{s_3,s_2,j} \Big(W(X^{s_3}Z^{s_2}J(\eta)J(\phi)E_{j,0}|R\rangle)_4 + W(X^{s_3}XZ^{s_2}J(\eta)J(-\phi)E_{j,1}|R\rangle)_4 \Big) \otimes m(s_2) \otimes m(s_3)$$
$$= \frac{1}{2^2 f_N(|L\rangle,|R\rangle)} \sum_{s_3,s_2,j} \Big(W(X^{s_3}Z^{s_2}J(\eta)J(\phi)E_{j,0}|R\rangle)_4 + W(X^{s_3}Z^{s_2}J(\eta)J(\phi)XE_{j,1}|R\rangle)_4 \Big) \otimes m(s_2) \otimes m(s_3).$$

In other words, the map

$$|R\rangle\langle R| \rightarrow \sum_j \left(E_{j,0}|R\rangle\langle R|E_{j,0}^\dagger + XE_{j,1}|R\rangle\langle R|E_{j,1}^\dagger X \right)$$

is implemented in the correlation space up to the rotation $J(\eta)J(\phi)$ and a Pauli byproduct $X^{s_3}Z^{s_2}$.

Since

$$\sum_j \left(E_{j,0}^\dagger E_{j,0} + (E_{j,1}^\dagger X)(XE_{j,1}) \right) = \sum_{j,s} E_{j,s}^\dagger E_{j,s} = I,$$

this is CPTP error. In short, a CPTP error on a physical qubit becomes a CPTP error in the correlation space of the cluster state. As is shown in Appendices A and B, similar result is obtained for the tricluster state [32], which is a variant of the cluster state.

III. AKLT STATE

We have seen in the previous section that CPTP errors on physical qubits of the one-dimensional cluster state become linear CPTP maps in the correlation space. However, this is not always the case for general resource states. In order to see it, let us consider the one-dimensional AKLT state as an example.

A. Simulation on the AKLT state without error

First we assume there is no error. The one-dimensional AKLT state is the matrix-product state defined by $d = 3$,

$$\begin{aligned} A[0] &= \frac{1}{\sqrt{3}}X, \\ A[1] &= \frac{1}{\sqrt{3}}XZ, \\ A[2] &= \frac{1}{\sqrt{3}}Z. \end{aligned}$$

If we measure the first physical qutrit of the AKLT state in the basis

$$\mathcal{M}_{\theta,\pi/2} \equiv \left\{ \cos\frac{\theta}{2}|0\rangle + i\sin\frac{\theta}{2}|1\rangle, \sin\frac{\theta}{2}|0\rangle - i\cos\frac{\theta}{2}|1\rangle, |2\rangle \right\},$$

we obtain

$$\frac{1}{3f_N(|L\rangle,|R\rangle)} \sum_{s_1=0}^{2} W(Q_1(s_1)|R\rangle)_2 \otimes \rho_1(s_1) \otimes m(s_1),$$

where $\rho_1(s_1)$ is the state of the first physical qutrit after the measurement, $m(s_1)$ is the register state which records the first measurement result, and

$$\begin{aligned} Q_1(0) &= XS_Z(\theta), \\ Q_1(1) &= XZS_Z(\theta), \\ Q_1(2) &= Z. \end{aligned}$$

Here, $S_Z(\theta) \equiv e^{-iZ\theta/2}$. By tracing out the first measured physical qutrit $\rho_1(s_1)$, we obtain

$$\frac{1}{3f_N(|L\rangle,|R\rangle)} \sum_{s_1=0}^{2} W(Q_1(s_1)|R\rangle)_2 \otimes m(s_1).$$

Next we measure the second physical qutrit by choosing measurement basis according to s_1 [29]. Then, we obtain

$$\frac{1}{3^2 f_N(|L\rangle,|R\rangle)} \sum_{s_2=0}^{2}\sum_{s_1=0}^{2} W(Q_2(s_1,s_2)Q_1(s_1)|R\rangle)_3 \otimes \rho_2(s_2,s_1) \otimes m(s_1) \otimes m(s_2),$$

where $\rho_2(s_1, s_2)$ is the state of the second physical qutrit after the measurement, $m(s_2)$ is the register state which records the second measurement result, and

$$Q_2(s_1, 0) = XS_Z(\theta),$$
$$Q_2(s_1, 1) = XZS_Z(\theta),$$
$$Q_2(s_1, 2) = Z,$$

if $s_1 = 2$ and

$$Q_2(s_1, 0) = X,$$
$$Q_2(s_1, 1) = XZ,$$
$$Q_2(s_1, 2) = Z,$$

for other s_1. By tracing out the measured second physical qutrit $\rho_2(s_2, s_1)$,

$$\frac{1}{3^2 f_N(|L\rangle, |R\rangle)} \sum_{s_2=0}^{2} \sum_{s_1=0}^{2} W(Q_2(s_1, s_2)Q_1(s_1)|R\rangle)_3 \otimes m(s_1) \otimes m(s_2).$$

If we repeat these process, after measuring the rth physical qutrit, we obtain

$$\frac{1}{3^r f_N(|L\rangle, |R\rangle)} \sum_{s_1=0}^{2} ... \sum_{s_r=0}^{2} W(Q_r(s_1, ..., s_r)...Q_2(s_1, s_2)Q_1(s_1)|R\rangle)_{r+1} \otimes m(s_1) \otimes ... \otimes m(s_r),$$

where

$$Q_k(s_1, ..., s_{k-1}, 0) = XS_Z(\theta),$$
$$Q_k(s_1, ..., s_{k-1}, 1) = XZS_Z(\theta),$$
$$Q_k(s_1, ..., s_{k-1}, 2) = Z,$$

if $s_1 = ... = s_{k-1} = 2$ and

$$Q_k(s_1, ..., s_{k-1}, 0) = X,$$
$$Q_k(s_1, ..., s_{k-1}, 1) = XZ,$$
$$Q_k(s_1, ..., s_{k-1}, 2) = Z,$$

for other $s_1, ..., s_{k-1}$.
   If $s_1 = ... = s_r = 2$,

$$Q_r(s_1, ..., s_r)...Q_2(s_1, s_2)Q_1(s_1) = Z^r.$$

For other $s_1, ..., s_r$,

$$Q_r(s_1, ..., s_r)...Q_2(s_1, s_2)Q_1(s_1) = X^{f(s_1,...,s_r)} Z^{g(s_1,...,s_r)} S_Z(\theta),$$

where

$$f(s_1, ..., s_r) = \bigoplus_{i=1}^{r} (\delta_{s_i,0} \oplus \delta_{s_i,1}),$$
$$g(s_1, ..., s_r) = \bigoplus_{i=1}^{r} (\delta_{s_i,1} \oplus \delta_{s_i,2}).$$

Let us add the flag state η as

$$\frac{1}{3^r f_N(|L\rangle, |R\rangle)} \sum_{s_1,...,s_r} W(Q_r(s_1, ..., s_r)...Q_2(s_1, s_2)Q_1(s_1)|R\rangle)_{r+1} \otimes m(s_1) \otimes ... \otimes m(s_r) \otimes \eta(f(s_1, ..., s_r), g(s_1, ..., s_r))$$

where $\eta(0,0)$, $\eta(0,1)$, $\eta(1,0)$, and $\eta(1,1)$, are mutually orthogonal with each other. If we trace out all register states, $m(s_1)$, $m(s_2)$, ..., and $m(s_r)$, we obtain

$$\frac{1}{3^r f_N(|L\rangle,|R\rangle)} \sum_{s_1,...,s_r} W(Q_r(s_1,...,s_r)...Q_2(s_1,s_2)Q_1(s_1)|R\rangle)_{r+1} \otimes \eta(f(s_1,...,s_r),g(s_1,...,s_r)) \tag{5}$$

$$= \frac{1}{3^r f_N(|L\rangle,|R\rangle)} \sum_{p=0}^{1} \sum_{q=0}^{1} \left(|S_{p,q}^r| \cdot W(X^p Z^q S_Z(\theta)|R\rangle)_{r+1} + h(p,q,r)W(Z^r|R\rangle)_{r+1} \right) \otimes \eta(p,q), \tag{6}$$

where

$$S_{p,q}^r \equiv \left\{ (s_1,...,s_r) \in \{0,1,2\}^{\times r} \setminus (2,...,2) \,\Big|\, f(s_1,...,s_r) = p \text{ and } g(s_1,...,s_r) = q \right\}$$

and

$$h(p,q,r) = \begin{cases} \delta_{p,0}\delta_{q,0} & (r = \text{even}) \\ \delta_{p,0}\delta_{q,1} & (r = \text{odd}). \end{cases}$$

In this way, we can implement the desired rotation $S_Z(\theta)$ up to Pauli byproducts $X^p Z^q$. Note that in the second term

$$h(p,q,r)W(Z^r|R\rangle_{r+1})$$

the desired rotation $S_Z(\theta)$ is not implemented (the trivial Z^r is implemented, instead). However, this term can be treated as the usual error which can be corrected by the usual FT circuits.

B.   Simulation on the AKLT state with error

Let us study what happens if an error occurs. We will see that not all physical errors on a physical qutrit can be linear CPTP errors in the correlation space.

If a CPTP error occurs on the first physical qutrit, Eq. (5) becomes

$$\frac{1}{3^{r-1} f_N(|L\rangle,|R\rangle)} \sum_{s_1,...,s_r} \sum_j W(Q_r(s_1,...,s_r)...Q_2(s_1,s_2)E_{j,s_1}|R\rangle)_{r+1} \otimes \eta(f(s_1,...,s_r),g(s_1,...,s_r))$$

$$= \frac{1}{3^{r-1} f_N(|L\rangle,|R\rangle)} \sum_j \sum_{p,q} \left(\sum_{(s_1,...,s_r)\in S_{p,q}^r} W(Q_r(s_1,...,s_r)...Q_2(s_1,s_2)E_{j,s_1}|R\rangle)_{r+1} \right.$$

$$\left. + h(p,q,r)W(Z^{r-1}E_{j,2}|R\rangle)_{r+1} \right) \otimes \eta(p,q).$$

This means that for fixed p and q, the map

$$|R\rangle\langle R| \;\rightarrow\; \sum_{(s_1,...,s_r)\in S_{p,q}^r} \sum_j Q_r(s_1,...,s_r)...Q_2(s_1,s_2)E_{j,s_1}|R\rangle\langle R|E_{j,s_1}^\dagger Q_2^\dagger(s_1,s_2)...Q_r^\dagger(s_1,...,s_r)$$

$$+ h(p,q,r)\sum_j Z^{r-1}E_{j,2}|R\rangle\langle R|E_{j,2}^\dagger Z^{r-1}$$

is implemented in the correlation space. Note that

$$\sum_{(s_1,...,s_r)\in S_{p,q}^r} \sum_j E_{j,s_1}^\dagger Q_2^\dagger(s_1,s_2)...Q_r^\dagger(s_1,...,s_r)Q_r(s_1,...,s_r)...Q_2(s_1,s_2)E_{j,s_1} + h(p,q,r)\sum_j E_{j,2}^\dagger Z^{r-1} Z^{r-1} E_{j,2}$$

$$= \sum_{(s_1,...,s_r)\in S_{p,q}^r} \sum_j E_{j,s_1}^\dagger E_{j,s_1} + h(p,q,r)\sum_j E_{j,2}^\dagger E_{j,2}$$

$$= \sum_{s_1} \sum_j |T_{p,q}^{r,s_1}| \cdot E_{j,s_1}^\dagger E_{j,s_1} + h(p,q,r)\sum_j E_{j,2}^\dagger E_{j,2}.$$

Here,

$$T^{r,i}_{p,q} \equiv \Big\{ (s_1, ..., s_r) \in \{0,1,2\}^{\times r} \setminus (2, ..., 2) \ \Big| \ s_1 = i \text{ and } f(s_1, ..., s_r) = p \text{ and } g(s_1, ..., s_r) = q \Big\}.$$

For example, let us consider the case $p = 1$ and $q = 0$. As is shown in Appendix E,

$$|T^{r,0}_{1,0}| - |T^{r,1}_{1,0}| \ = \ (-1)^{r-1},$$
$$|T^{r,1}_{1,0}| \ = \ |T^{r,2}_{1,0}|.$$

Therefore,

$$\sum_{s_1} \sum_{j=1}^{w} |T^{r,s_1}_{1,0}| \cdot E^{\dagger}_{j,s_1} E_{j,s_1} = \begin{cases} |T^{r,1}_{1,0}| I + \sum_j E^{\dagger}_{j,0} E_{j,0} & (r = \text{odd}) \\ |T^{r,0}_{1,0}| I + \sum_j E^{\dagger}_{j,1} E_{j,1} + \sum_j E^{\dagger}_{j,2} E_{j,2} & (r = \text{even}). \end{cases}$$

For example, if $w = 1$ and

$$F_1 = U_{\mathcal{M}_{\theta,\phi}} \Big(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 0| - \frac{|0\rangle - |1\rangle}{\sqrt{2}} \langle 1| + |2\rangle \langle 2| \Big),$$

$$E_{1,0} = \sqrt{\frac{2}{3}} |0\rangle \langle 1|$$

$$E_{1,1} = \sqrt{\frac{2}{3}} |1\rangle \langle 0|$$

$$E_{1,2} = \frac{1}{\sqrt{3}} Z.$$

Then,

$$\sum_{s_1} \sum_{j=1}^{w} |T^{r,s_1}_{1,0}| \cdot E^{\dagger}_{j,s_1} E_{j,s_1} = \begin{cases} |T^{r,1}_{1,0}| I + \frac{2}{3} |1\rangle \langle 1| & (r = \text{odd}) \\ |T^{r,0}_{1,0}| I + \frac{2}{3} |0\rangle \langle 0| + \frac{1}{3} I & (r = \text{even}), \end{cases} \tag{7}$$

which means that the map implemented in the correlation space is not linear CPTP.

IV. INTUITIVE EXPLANATIONS

So far, we have seen that the results of Refs. [18, 19] for the cluster state cannot be directly applied to other resource states, such as the one-dimensional AKLT state. Why the cluster state is so special? And why direct applications of the result for the cluster state to other resource states do not work? Although the complete answer to these questions is beyond the scope of the present paper, since the study of QCTN itself has not been fully developed (for example, no one knows the necessary and sufficient condition for tensor-network states to be universal resource states for measurement-based quantum computation), let us try to give some intuitive explanations here.

Figure 1 illustrates the reason why all physical errors become CPTP maps in the correlation space of the one-dimensional cluster state. Let us first consider the ideal case (a) where there is no error. The actual protocol (a-1) is mathematically equivalent to the "input-output" picture (a-2) where the physical input state $|\psi\rangle$ is teleported into the left-edge of the short-length chain (indicated in yellow) and finally the physical output state $|\psi'\rangle$ is extracted from the right-edge of the short-length chain (indicated in yellow). Since both $|\psi\rangle$ and $|\psi'\rangle$ are physical states, what is going on in the correlation space which maps the input state $|\psi\rangle$ to the output state $|\psi'\rangle$ can be described by a linear CPTP operation. In other words, if we can describe measurement-based quantum computation with this "input-output" picture [19], the map implemented in the correlation space is guaranteed to be a linear CPTP operation [19].

For the cluster state, this "input-output" picture also holds even if there is an error [19]: In the imperfect case, Fig. 1 (b), let us assume that the input state is degraded by an error and becomes a mixed state ρ. However, we can still consider a similar "input-output" picture (b-2), which corresponds to the actual protocol (b-1), and again the physical state ρ is mapped into another physical state ρ', which means that what is going on in the correlation space which maps ρ to ρ' can be described by a linear CPTP operation.

Note that two special properties of the cluster state enable such an "input-output" picture. First, the one-dimensional cluster state can be decomposed into small pieces of one-dimensional cluster states by applying nearest-neighbour two-body unitary operations (i.e., CZ gates). As is shown in Fig. 1 (a-2) and (b-2), this property is necessary for allowing the "input-output" picture. Second, the number of qubits that are measured in order to implement a specific gate does not depend on the measurement results. In other words, for the cluster state, a specific gate can be implemented up to Pauli byproducts at a *fixed* site irrespective of measurement results. Such a deterministic implementation at a fixed site is necessary for the deterministic (i.e., trace-preserving) "output" in the "input-output" picture, since in the method of Refs. [19], the ensemble (mixture) of all measurement results are considered: If the site where the desired gate operation is completed depends on the measurement results, we cannot "extract" *the same* output state at a *fixed* site irrespective of measurement results as is shown in Fig. 1 (a-2) and (b-2).

On the other hand, such an "input-output" picture seems to be impossible for the one-dimensional AKLT state, because of the following two reasons: First, as is shown in Fig. 2 left, no nearest-neighbour two-body unitary operation can decompose the one-dimensional AKLT chain into two chains due to the existence of the non-vanishing two-point correlation in the AKLT state. (If the one-dimensional AKLT chain can be decomposed into two chains by such a unitary, it contradicts to the well-known fact that the two-point correlation is non-vanishing in the AKLT state.) Second, we cannot deterministically implement a specific gate at a *fixed* site of the AKLT chain irrespective of measurement results [4, 5, 29]. In short, the "input-output" picture seems to be impossible for the AKLT state. If we can no longer use the "input-output" picture, it is not unreasonable that we have some anomalous maps in the correlation space since the correlation space is not a physical space but an abstract mathematical space.
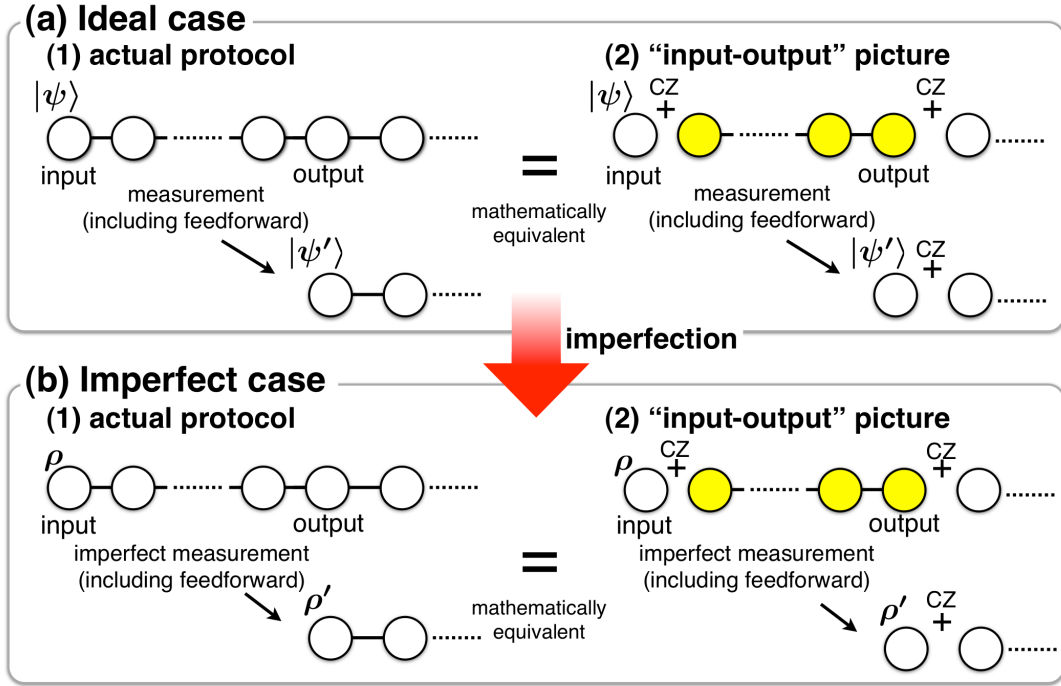

FIG. 1: The "input-output" picture for the one-dimensional cluster state.

V. ANOTHER WAY OF SIMULATING QUANTUM CIRCUITS

As we have seen in the previous section, it is not always possible for general resource states to implement a specific gate at a fixed site of the resource state irrespective of measurement results. This fact prohibits a general resource state from allowing the "input-output" picture. One might think that if we abandon such a deterministic "output" at a fixed site for all measurement results, and if we just consider a specific history (trajectory) of measurement results, we might be able to avoid the emergence of non-CPTP errors. (Physically, this means that we project the system onto a pure state at every measurement step.) If the system is assumed to be error-free, this "trajectory method" is another standard way of simulating quantum circuits in the correlation space [4–6, 10, 29]. (Note that
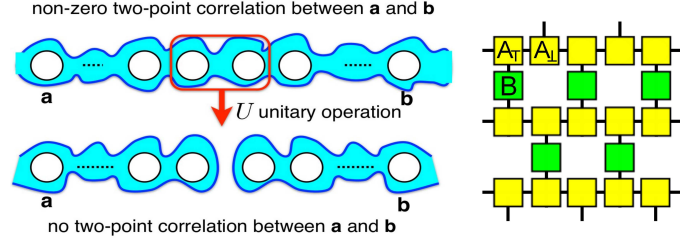
FIG. 2: Left: The one-dimensional AKLT state cannot be decomposed into short chains. Right: The tensor network for the resource state in Ref. [31].

in this trajectory method, correct unitary operators can be implemented in the correlation space if there is no error, although what we are physically doing are projections, i.e., non-trace-preserving operations).

However, we here show that such a natural another way of simulating quantum circuits does neither work if $d \geq 3$. In other words, we can show the following theorem.

A. Theorem

Theorem: If $d \geq 3$, there exists a single-qudit CPTP error $\mathcal{E}$ which has the following property: assume that $\mathcal{E}$ is applied on a single physical qudit of $|\Psi(L, R)\rangle_1^N$. If the measurement $\mathcal{M}_{\theta,\phi}$, Eq. (2), is performed on that affected qudit, a non-TP operation is implemented in the correlation space.

Proof: In order to show Theorem, let us assume that

$$\text{There is no such } \mathcal{E}. \tag{8}$$

We will see that this assumption leads to the contradiction that $d \leq 2$.

First, let us consider the state

$$(I^{\otimes N-1} \otimes U_{1 \leftrightarrow 2})|\Psi(L, R)\rangle_1^N, \tag{9}$$

where

$$U_{a \leftrightarrow b} \equiv |a\rangle\langle b| + |b\rangle\langle a| + I - |a\rangle\langle a| - |b\rangle\langle b|$$

is the unitary error which exchanges $|a\rangle$ and $|b\rangle$, and I is the identity operator on a single qudit. In Eq. (9), the error $U_{1 \leftrightarrow 2}$ is applied on the first physical qudit of $|\Psi(L, R)\rangle_1^N$. If we do the measurement $\mathcal{M}_{\theta,\phi}$ on the first physical qudit of Eq. (9), and if the measurement result is $|2\rangle$, Eq. (9) becomes

$$\frac{1}{\sqrt{f_{N-1}(|L\rangle, A[1]|R\rangle)}} \|A[1]\| \sum_{k_2=0}^{d-1} ... \sum_{k_N=0}^{d-1} \langle L|A[k_N]...A[k_2] \frac{A[1]}{\|A[1]\|}|R\rangle|k_N, ..., k_2\rangle \otimes |2\rangle. \tag{10}$$

In other words, the operation

$$|R\rangle \to \frac{A[1]}{\|A[1]\|}|R\rangle$$

is implemented in the correlation space. By the assumption Eq. (8), this operation should work as a TP operation in the correlation space. Therefore,

$$\frac{A^\dagger[1]}{\|A[1]\|} \frac{A[1]}{\|A[1]\|} = I. \tag{11}$$

By taking $\eta \equiv \|A[1]\|^2$,

$$A^\dagger[1]A[1] = \eta I. \tag{12}$$

Second, let us consider the measurement $\mathcal{M}_{\theta,\phi}$ on the first physical qudit of

$$(I^{\otimes N-1} \otimes U_{0\leftrightarrow 2} V^s)|\Psi(L,R)\rangle_1^N,$$

where $s \in \{0, 1, ..., d-1\}$,

$$V \equiv \sum_{p=0}^{d-1} e^{-i\omega p}|p\rangle\langle p|$$

is a unitary phase error, and $\omega \equiv 2\pi/d$. If the measurement result is $|\alpha_{\theta,\phi}\rangle$,

$$\left(e^{-2is\omega}\cos\frac{\theta}{2}A[2] + e^{-i(\phi+s\omega)}\sin\frac{\theta}{2}A[1]\right)/\sqrt{\gamma}$$

is implemented in the correlation space, where

$$\sqrt{\gamma} \equiv \left\|e^{-2is\omega}\cos\frac{\theta}{2}A[2] + e^{-i(\phi+s\omega)}\sin\frac{\theta}{2}A[1]\right\|.$$

By the assumption Eq. (8), this should work as a TP operation in the correlation space. Therefore,

$$\gamma I = \cos^2\frac{\theta}{2}A^\dagger[2]A[2] + \sin^2\frac{\theta}{2}A^\dagger[1]A[1] + \frac{1}{2}\sin\theta\left(e^{-i(\phi-s\omega)}A^\dagger[2]A[1] + e^{i(\phi-s\omega)}A^\dagger[1]A[2]\right).$$

By the assumption Eq. (3),

$$A^\dagger[2]A[2] = \xi I,$$

where $\xi \equiv \|A[2]\|^2$. Furthermore, as we have shown, $A^\dagger[1]A[1] = \eta I$ (Eq. (12)). Therefore,

$$\gamma' I = e^{-i(\phi-s\omega)}A^\dagger[2]A[1] + e^{i(\phi-s\omega)}A^\dagger[1]A[2], \tag{13}$$

where

$$\gamma' \equiv \frac{2}{\sin\theta}\left(\gamma - \xi\cos^2\frac{\theta}{2} - \eta\sin^2\frac{\theta}{2}\right).$$

Finally, let us consider the measurement $\mathcal{M}_{\theta,\phi}$ on the first physical qudit of

$$(I^{\otimes N-1} \otimes U_{0\leftrightarrow 1}U_{0\leftrightarrow 2}V^t)|\Psi(L,R)\rangle_1^N,$$

where $t \in \{0, 1, ..., d-1\}$. If the measurement result is $|\alpha_{\theta,\phi}\rangle$,

$$\left(e^{-it\omega}\cos\frac{\theta}{2}A[1] + e^{-i\phi-2it\omega}\sin\frac{\theta}{2}A[2]\right)/\sqrt{\delta}$$

is implemented in the correlation space, where

$$\sqrt{\delta} \equiv \left\|e^{-it\omega}\cos\frac{\theta}{2}A[1] + e^{-i\phi-2it\omega}\sin\frac{\theta}{2}A[2]\right\|.$$

By the assumption Eq. (8), this should also work as a TP operation in the correlation space. Therefore,

$$\delta' I = e^{i(\phi+t\omega)}A^\dagger[2]A[1] + e^{-i(\phi+t\omega)}A^\dagger[1]A[2], \tag{14}$$

where

$$\delta' \equiv \frac{2}{\sin\theta}\left(\delta - \xi\sin^2\frac{\theta}{2} - \eta\cos^2\frac{\theta}{2}\right).$$

From Eqs. (13) and (14),

$$\epsilon I = \left[e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)}\right]A^\dagger[2]A[1],$$

where

$$\epsilon \equiv e^{-i(\phi-s\omega)}\gamma' - e^{i(\phi+t\omega)}\delta'.$$

Let us assume that

$$e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)} \neq 0.$$

Then,

$$\epsilon' I = A^\dagger[2]A[1],$$

where

$$\epsilon' \equiv \frac{\epsilon}{e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)}}.$$

If $\epsilon' = 0$, $A^\dagger[2]A[1] = 0$, which means $A[1] = 0$ since $A[2]$ is unitary up to a constant (assumption Eq. (3)). Therefore, $\epsilon' \neq 0$. In this case, $A[1] = \epsilon''A[2]$ for certain $\epsilon'' \neq 0$, since $A[2]$ is unitary up to a constant [34]. Hence

$$e^{-2i(\phi-s\omega)} - e^{2i(\phi+t\omega)} = 0.$$

This means

$$2\phi + (t-s)\omega = r_{s,t}\pi, \tag{15}$$

where $r_{s,t} \in \{0,1,2,3,...\}$. Let us take $t = s = 0$. Then, Eq. (15) gives

$$\phi = r_{0,0}\frac{\pi}{2} \quad (r_{0,0} \in \{0,1,2,...\}).$$

Let us take $s = 1$, $t = 0$. Then, Eq. (15) gives

$$\phi = \frac{\pi}{d} + r_{1,0}\frac{\pi}{2} \quad (r_{1,0} \in \{0,1,2,...\}).$$

In order to satisfy these two equations at the same time, there must exist $r_{0,0}$ and $r_{1,0}$ such that

$$r_{0,0}\frac{\pi}{2} = \frac{\pi}{d} + r_{1,0}\frac{\pi}{2}.$$

If $r_{0,0} = r_{1,0}$, then $0 = 1/d$ which means $d = \infty$. Therefore $r_{0,0} \neq r_{1,0}$. Then we have

$$d = \frac{2}{r_{0,0} - r_{1,0}} \leq 2,$$

which is the contradiction. ∎

One might think that if we rewrite the post-measurement state Eq. (10) as

$$\frac{1}{\sqrt{f_{N-1}(|L\rangle, A[1]|R\rangle)}} \| A[1]|R\rangle\| \sum_{k_2=0}^{d-1} ... \sum_{k_N=0}^{d-1} \langle L|A[k_N]...A[k_2]\frac{A[1]}{\|A[1]|R\rangle\|}|R\rangle|k_N, ..., K_2\rangle \otimes |2\rangle$$

and redefine the operation implemented in the correlation space as

$$|R\rangle \rightarrow \frac{A[1]}{\|A[1]|R\rangle\|}|R\rangle,$$

the TP-ness is recovered in the correlation space. However, in this case, the non-lineally appears unless

$$A^\dagger[1]A[1] \propto I,$$

and therefore if we require the linearity in the correlation space, we obtain the same contradiction.

In short, if $d \geq 3$ not all physical errors on physical qudits appear as linear CPTP errors in the correlation space of pure matrix product states [35].

B. Examples

Let us consider some concrete examples which give intuitive understandings of the above theorem.

In Ref. [29], it was shown that universal single-qubit unitary rotation is possible in the correlation space of the one-dimensional AKLT chain, which is a ground state of a gapped two-body nearest-neighbour spin-1 Hamiltonian (hence $d = 3$). The matrix product representation of the one-dimensional AKLT chain is given by

$$
\begin{aligned}
A[0] &= X, \\
A[1] &= XZ, \\
A[2] &= Z
\end{aligned}
$$

for certain basis $\{|0\rangle, |1\rangle, |2\rangle\}$ [29]. The measurement

$$
\mathcal{M}_{\theta, \pi/2} \equiv \left\{ \cos\frac{\theta}{2}|0\rangle + i\sin\frac{\theta}{2}|1\rangle, \sin\frac{\theta}{2}|0\rangle - i\cos\frac{\theta}{2}|1\rangle, |2\rangle \right\}
$$

on a single physical qutrit implements $Xe^{-iZ\theta/2}$, $XZe^{-iZ\theta/2}$, or Z, respectively. According to Theorem in the previous section, not all physical errors can be linear CPTP maps in the correlation space since $d \geq 3$. In fact, let us consider the single qutrit unitary error

$$
U \equiv |2\rangle \frac{\langle 0| + \langle 1|}{\sqrt{2}} + \frac{|0\rangle + |1\rangle}{\sqrt{2}} \langle 2| + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \frac{\langle 0| - \langle 1|}{\sqrt{2}}.
$$

If the measurement $\mathcal{M}_{\theta,\pi/2}$ is performed after the error U and if we obtain the result $|2\rangle$, the operation $|1\rangle\langle 0|$ is implemented in the correlation space. Obviously, it is not TP. The same result is obtained for the slightly modified version of the one-dimensional AKLT chain [4], where

$$
\begin{aligned}
A[0] &= X, \\
A[1] &= XZ, \\
A[2] &= H.
\end{aligned}
$$

In Ref. [31], it was shown that the unique ground state of a gapped two-body nearest-neighbour spin-3/2 Hamiltonian with the AKLT and exchange interactions on the two-dimensional octagonal lattice is a universal resource state for the measurement-based quantum computation. The state is defined by the following tensor network [31]:

$$
\begin{aligned}
A_\top\left[+\frac{3}{2}\right] &= |1\rangle\langle 0| \otimes \langle 1|, \\
A_\top\left[-\frac{3}{2}\right] &= |0\rangle\langle 1| \otimes \langle 0|, \\
A_\top\left[+\frac{1}{2}\right] &= -\frac{1}{\sqrt{3}}\Big(Z \otimes \langle 1| + |1\rangle\langle 0| \otimes \langle 0|\Big), \\
A_\top\left[-\frac{1}{2}\right] &= \frac{1}{\sqrt{3}}\Big(Z \otimes \langle 0| - |0\rangle\langle 1| \otimes \langle 1|\Big), \\
B\left[+\frac{3}{2}\right] &= |0\rangle\langle 1|, \\
B\left[-\frac{3}{2}\right] &= -|1\rangle\langle 0|, \\
B\left[+\frac{1}{2}\right] &= |1\rangle\langle 1|, \\
B\left[-\frac{1}{2}\right] &= -|0\rangle\langle 0|,
\end{aligned}
$$

(see Fig. 2 right). $A_\perp$'s are defined in the same way. Each horizontal line works as a single-qubit wire. Two nearest-neighbour horizontal chains are decoupled by measuring sites B in the z-basis. Before starting the computation, the filtering operation $\{F, F'\}$, where

$$
\begin{aligned}
F &\equiv \frac{1}{\sqrt{3}}|3/2\rangle\langle 3/2| + \frac{1}{\sqrt{3}}|-3/2\rangle\langle -3/2| + |1/2\rangle\langle 1/2| + |-1/2\rangle\langle -1/2| \\
F' &\equiv \sqrt{\frac{2}{3}}|3/2\rangle\langle 3/2| + \sqrt{\frac{2}{3}}|-3/2\rangle\langle -3/2|,
\end{aligned}
$$

is applied on each site A. Let us assume that the filtering is succeeded (i.e., F is realized) and a site B is projected onto $|3/2\rangle$. Then, the measurement

$$\mathcal{M}_{\pi/2,\phi} \equiv \left\{ \frac{1}{\sqrt{2}}\Big(|1/2\rangle \pm e^{i\phi}|-3/2\rangle\Big), |-1/2\rangle, |+3/2\rangle \right\}$$

implements $ZXe^{iZ\phi/2}$, $Xe^{iZ\phi/2}$, or Z, respectively. (The result $|3/2\rangle$ does not occur.) If the error which exchanges $|1/2\rangle$ and $|-1/2\rangle$ occurs and if the result of the measurement $\mathcal{M}_{\pi/2,\phi}$ is $|-1/2\rangle$, the operation $|1\rangle\langle 0|$, which is not TP, is implemented in the correlation space.

VI.   CONCLUSION

In this paper, we have studied how physical errors on a physical qudit appear in the correlation space of general resource states. We have shown that the results [18, 19] for the cluster state cannot be directly applied to general resource states, such the AKLT state. We have also shown that if $d \geq 3$ not all physical errors can be linear CPTP errors in the correlation space of pure matrix product states. These results suggest that the application of the theories of fault-tolerant quantum circuits to the correlation space of general resource states is not so straightforward.

Acknowledgments

Appendix A: Tricluster state without error

Let us consider the tricluster state [32], where $d = 6$ and

$$
\begin{aligned}
A[0] &= |+\rangle\langle 0|, \\
A[1] &= |-\rangle\langle 1|, \\
A[2] &= |-\rangle\langle 0|, \\
A[3] &= |+\rangle\langle 1|, \\
A[4] &= |+\rangle\langle 1|, \\
A[5] &= |-\rangle\langle 0|.
\end{aligned}
$$

The measurement

$$
\begin{aligned}
|\theta_0\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\theta}|1\rangle\Big) \\
|\theta_1\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|0\rangle - e^{i\theta}|1\rangle\Big) \\
|\theta_2\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|2\rangle + e^{i\theta}|3\rangle\Big) \\
|\theta_3\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|2\rangle - e^{i\theta}|3\rangle\Big) \\
|\theta_4\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|4\rangle + e^{-i\theta}|5\rangle\Big) \\
|\theta_5\rangle &\equiv \frac{1}{\sqrt{2}}\Big(|4\rangle - e^{-i\theta}|5\rangle\Big)
\end{aligned}
$$

is performed in order to implement

$$
\begin{aligned}
|\theta_0\rangle &: J(\theta) \\
|\theta_1\rangle &: XJ(\theta) \\
|\theta_2\rangle &: ZJ(\theta) \\
|\theta_3\rangle &: ZXJ(\theta) \\
|\theta_4\rangle &: ZJ(\theta) \\
|\theta_5\rangle &: ZXJ(\theta).
\end{aligned}
$$

First, we measure the first physical qudit in the $\{|\theta_{s_1}\rangle\}$ basis. Then we obtain

$$
\frac{1}{2 f_N(|L\rangle, |R\rangle)} \sum_{s_1=0}^{5} W(X^{p(s_1)} Z^{q(s_1)} J(\theta)|R\rangle)_2 \otimes |\theta_{s_1}\rangle\langle\theta_{s_1}| \otimes m(s_1),
$$

where

$$
\begin{aligned}
p(s) &\equiv \delta_{s,1} + \delta_{s,3} + \delta_{s,5} \\
q(s) &\equiv \delta_{s,2} + \delta_{s,3} + \delta_{s,4} + \delta_{s,5}.
\end{aligned}
$$

If we trace out the first physical qudit, we obtain

$$
\frac{1}{2 f_N(|L\rangle, |R\rangle)} \sum_{s_1=0}^{5} W(X^{p(s_1)} Z^{q(s_1)} J(\theta)|R\rangle)_2 \otimes m(s_1).
$$

Second, if we measure the second physical qudit in the basis

$$
\begin{aligned}
|s_1, \phi_0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{q(s_1)} e^{i\phi(-1)^{p(s_1)}} |1\rangle\right) \\
|s_1, \phi_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - (-1)^{q(s_1)} e^{i\phi(-1)^{p(s_1)}} |1\rangle\right) \\
|s_1, \phi_2\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle + (-1)^{q(s_1)} e^{i\phi(-1)^{p(s_1)}} |3\rangle\right) \\
|s_1, \phi_3\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle - (-1)^{q(s_1)} e^{i\phi(-1)^{p(s_1)}} |3\rangle\right) \\
|s_1, \phi_4\rangle &= \frac{1}{\sqrt{2}}\left(|4\rangle + (-1)^{q(s_1)} e^{i\phi(-1)^{p(s_1)}} |5\rangle\right) \\
|s_1, \phi_5\rangle &= \frac{1}{\sqrt{2}}\left(|4\rangle - (-1)^{q(s_1)} e^{i\phi(-1)^{p(s_1)}} |5\rangle\right)
\end{aligned}
\tag{A1}
$$

and trace out the measured qudit, we obtain

$$
\frac{1}{2^2 f_N(|L\rangle, |R\rangle)} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} W(X^{p(s_2)} Z^{q(s_2)} X^{q(s_1)} J((-1)^{p(s_1)}\phi) X^{p(s_1)} Z^{q(s_1)} J(\theta)|R\rangle)_3 \otimes m(s_1) \otimes m(s_2)
$$

$$
= \frac{1}{2^2 f_N(|L\rangle, |R\rangle)} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} W(X^{p(s_2)} Z^{q(s_2)} Z^{p(s_1)} J(\phi) J(\theta)|R\rangle)_3 \otimes m(s_1) \otimes m(s_2).
$$

Third, if we measure the third physical qudit in the basis

$$
\begin{aligned}
|s_1, s_2, \eta_0\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{p(s_1)+q(s_2)} e^{i\eta(-1)^{p(s_2)}}|1\rangle\right) \\
|s_1, s_2, \eta_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - (-1)^{p(s_1)+q(s_2)} e^{i\eta(-1)^{p(s_2)}}|1\rangle\right) \\
|s_1, s_2, \eta_3\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle + (-1)^{p(s_1)+q(s_2)} e^{i\eta(-1)^{p(s_2)}}|3\rangle\right) \\
|s_1, s_2, \eta_4\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle - (-1)^{p(s_1)+q(s_2)} e^{i\eta(-1)^{p(s_2)}}|3\rangle\right) \\
|s_1, s_2, \eta_5\rangle &= \frac{1}{\sqrt{2}}\left(|4\rangle + (-1)^{p(s_1)+q(s_2)} e^{i\eta(-1)^{p(s_2)}}|5\rangle\right) \\
|s_1, s_2, \eta_6\rangle &= \frac{1}{\sqrt{2}}\left(|4\rangle - (-1)^{p(s_1)+q(s_2)} e^{i\eta(-1)^{p(s_2)}}|5\rangle\right)
\end{aligned}
\tag{A2}
$$

and trace out the measured third qudit, we obtain

$$
\begin{aligned}
&\frac{1}{2^3 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{5} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} \\
&W(X^{p(s_3)} Z^{q(s_3)} X^{p(s_1)+q(s_2)} J((-1)^{p(s_2)}\eta) X^{p(s_2)} Z^{q(s_2)} Z^{p(s_1)} J(\phi) J(\theta)|R\rangle)_4 \otimes m(s_1) \otimes m(s_2) \otimes m(s_3) \\
&= \frac{1}{2^3 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{5} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} W(X^{p(s_3)} Z^{q(s_3)} Z^{p(s_2)} J(\eta) J(\phi) J(\theta)|R\rangle)_4 \otimes m(s_1) \otimes m(s_2) \otimes m(s_3) \\
&= \frac{1}{2^3 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{5} \sum_{s_2=0}^{5} W(X^{p(s_3)} Z^{q(s_3)} Z^{p(s_2)} J(\eta) J(\phi) J(\theta)|R\rangle)_4 \otimes m(s_2) \otimes m(s_3).
\end{aligned}
$$

Appendix B: Tricluster state with error

Let us assume that a CPTP error occurs on the first physical qudit. If we measure the first physical qudit in the $\{|\theta_{s_1}\rangle\}$ basis, we obtain

$$
\frac{1}{f_N(|L\rangle, |R\rangle)} \sum_{s_1=0}^{5} \sum_j W(E_{j,s_1}|R\rangle)_2 \otimes |\theta_{s_1}\rangle\langle\theta_{s_1}| \otimes m(s_1).
$$

By tracing out the first physical qudit, we obtain

$$
\frac{1}{f_N(|L\rangle, |R\rangle)} \sum_{s_1=0}^{5} \sum_j W(E_{j,s_1}|R\rangle)_2 \otimes m(s_1).
$$

Second, if we measure the second physical qudit in the basis Eq. (A1) and trace out the measured qudit, we obtain

$$
\frac{1}{2 f_N(|L\rangle, |R\rangle)} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} \sum_j W(X^{p(s_2)} Z^{q(s_2)} X^{q(s_1)} J((-1)^{p(s_1)}\phi) E_{j,s_1}|R\rangle)_3 \otimes m(s_1) \otimes m(s_2).
$$

Third, if we measure the third physical qudit in the basis Eq. (A2) and trace out the measured third qudit, we obtain

$$
\begin{aligned}
&\frac{1}{2^2 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{5} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} \sum_j \\
&W(X^{p(s_3)} Z^{q(s_3)} X^{p(s_1)+q(s_2)} J((-1)^{p(s_2)}\eta) X^{p(s_2)} Z^{q(s_2)} X^{q(s_1)} J((-1)^{p(s_1)}\phi) E_{j,s_1}|R\rangle)_4 \otimes m(s_1) \otimes m(s_2) \otimes m(s_3) \\
&= \frac{1}{2^2 f_N(|L\rangle, |R\rangle)} \sum_{s_3=0}^{5} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} \sum_j \\
&W(X^{p(s_3)} Z^{q(s_3)} Z^{p(s_2)} X^{p(s_1)} J(\eta) X^{q(s_1)} J((-1)^{p(s_1)}\phi) E_{j,s_1}|R\rangle)_4 \otimes m(s_1) \otimes m(s_2) \otimes m(s_3).
\end{aligned}
$$

If we trace out the first record,

$$\frac{1}{2^2 f_N(|L\rangle,|R\rangle)} \sum_{s_3=0}^{5} \sum_{s_2=0}^{5} \sum_{s_1=0}^{5} \sum_{j} W(X^{p(s_3)} Z^{q(s_3)} Z^{p(s_2)} X^{p(s_1)} J(\eta) X^{q(s_1)} J((-1)^{p(s_1)}\phi) E_{j,s_1} |R\rangle)_4 \otimes m(s_2) \otimes m(s_3).$$

Thus the map

$$|R\rangle\langle R| \to \sum_{s_1=0}^{5} \sum_{j} X^{p(s_1)} J(\eta) X^{q(s_1)} J((-1)^{p(s_1)}\phi) E_{j,s_1} |R\rangle\langle R| E_{j,s_1}^{\dagger} J^{\dagger}((-1)^{p(s_1)}\phi) X^{q(s_1)} J^{\dagger}(\eta) X^{p(s_1)},$$

which is obviously TP, is implemented.

<h2 style="text-align:center">Appendix C: Calculation of $|U_{p,q}^r|$</h2>

Let us define

$$U_{p,q}^r \equiv \left\{ (s_1,...,s_r) \in \{0,1,2\}^{\times r} \;\middle|\; f(s_1,...,s_r) = p \text{ and } g(s_1,...,s_r) = q \right\}.$$

First,

$$\begin{aligned}
|U_{0,0}^2| &= 3 \\
|U_{0,1}^2| &= 2 \\
|U_{1,0}^2| &= 2 \\
|U_{1,1}^2| &= 2.
\end{aligned}$$

Second,

$$|U_{p,q}^r| = |U_{p\oplus 1,q}^{r-1}| + |U_{p,q\oplus 1}^{r-1}| + |U_{p\oplus 1,q\oplus 1}^{r-1}|$$

for all r. Therefore,

$$\begin{aligned}
|U_{0,1}^r| - |U_{1,0}^r| &= |U_{1,1}^{r-1}| + |U_{0,0}^{r-1}| + |U_{1,0}^{r-1}| - |U_{0,0}^{r-1}| - |U_{1,1}^{r-1}| - |U_{0,1}^{r-1}| \\
&= |U_{1,0}^{r-1}| - |U_{0,1}^{r-1}| \\
&= -(|U_{0,1}^{r-1}| - |U_{1,0}^{r-1}|) \\
&= (-1)^{r-2}(|U_{0,1}^2| - |U_{1,0}^2|) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
|U_{0,1}^r| - |U_{1,1}^r| &= |U_{1,1}^{r-1}| + |U_{0,0}^{r-1}| + |U_{1,0}^{r-1}| - |U_{0,1}^{r-1}| - |U_{1,0}^{r-1}| - |U_{0,0}^{r-1}| \\
&= |U_{1,1}^{r-1}| - |U_{0,1}^{r-1}| \\
&= -(|U_{0,1}^{r-1}| - |U_{1,1}^{r-1}|) \\
&= (-1)^{r-2}(|U_{0,1}^2| - |U_{1,1}^2|) \\
&= 0,
\end{aligned}$$

which mean

$$|U_{1,1}^r| = |U_{0,1}^r| = |U_{1,0}^r|$$

for all r.

Note that

$$\begin{aligned}
|U_{0,0}^r| - |U_{0,1}^r| &= 3|U_{0,1}^{r-1}| - (2|U_{0,1}^{r-1}| + |U_{0,0}^{r-1}|) \\
&= |U_{0,1}^{r-1}| - |U_{0,0}^{r-1}| \\
&= -(|U_{0,0}^{r-1}| - |U_{0,1}^{r-1}|) \\
&= (-1)^{r-2}(|U_{0,0}^2| - |U_{0,1}^2|) \\
&= (-1)^{r-2}.
\end{aligned}$$

and

$$
\begin{aligned}
|U_{0,0}^r| + 3|U_{0,1}^r| &= 3|U_{0,1}^{r-1}| + 3(2|U_{0,1}^{r-1}| + |U_{0,0}^{r-1}|) \\
&= 3|U_{0,0}^{r-1}| + 9|U_{0,1}^{r-1}| \\
&= 3(|U_{0,0}^{r-1}| + 3|U_{0,1}^{r-1}|) \\
&= 3^{r-2}(|U_{0,0}^2| + 3|U_{0,1}^2|) \\
&= 3^r.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|U_{0,1}^r| &= \frac{1}{4}(3^r - (-1)^r) \\
|U_{0,0}^r| &= \frac{1}{4}(3^r + 3(-1)^r).
\end{aligned}
$$

Appendix D: Calculation of $|S_{p,q}^r|$

Let us define

$$
S_{p,q}^r \equiv \left\{ (s_1, ..., s_r) \in \{0,1,2\}^{\times r} \setminus (2,...,2) \,\middle|\, f(s_1,...,s_r) = p \text{ and } g(s_1,...,s_r) = q \right\}.
$$

First,

$$
\begin{aligned}
|S_{0,0}^2| &= 2 \\
|S_{0,1}^2| &= 2 \\
|S_{1,0}^2| &= 2 \\
|S_{1,1}^2| &= 2.
\end{aligned}
$$

Second,

$$
|S_{p,q}^r| = |U_{p\oplus 1,q}^{r-1}| + |U_{p\oplus 1,q\oplus 1}^{r-1}| + |S_{p,q\oplus 1}^{r-1}|.
$$

Therefore,

$$
\begin{aligned}
|S_{0,0}^r| - |S_{0,1}^r| &= |U_{1,0}^{r-1}| + |U_{1,1}^{r-1}| + |S_{0,1}^{r-1}| - |U_{1,1}^{r-1}| - |U_{1,0}^{r-1}| - |S_{0,0}^{r-1}| \\
&= -(|S_{0,0}^{r-1}| - |S_{0,1}^{r-1}|) \\
&= (-1)^{r-2}(|S_{0,0}^2| - |S_{0,1}^2|) \\
&= 0.
\end{aligned}
$$

and

$$
\begin{aligned}
|S_{1,0}^r| - |S_{1,1}^r| &= |U_{0,0}^{r-1}| + |U_{0,1}^{r-1}| + |S_{1,1}^{r-1}| - |U_{0,1}^{r-1}| - |U_{0,0}^{r-1}| - |S_{1,0}^{r-1}| \\
&= -(|S_{1,0}^{r-1}| - |S_{1,1}^{r-1}|) \\
&= (-1)^{r-2}(|S_{1,0}^2| - |S_{1,1}^2|) \\
&= 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
|S_{0,0}^r| &= \frac{3^{r-1} - (-1)^{r-1}}{2} + |S_{0,0}^{r-1}| \\
&= \frac{3^{r-1} - (-1)^{r-1}}{2} + ... + \frac{3^2 - (-1)^2}{2} + |S_{0,0}^2| \\
&= \frac{3^{r-1} + ... + 3^2}{2} - \frac{(-1)^{r-1} + ... + (-1)^2}{2} + 2 \\
&= \frac{3^r - 3^2}{4} + \frac{(-1)^r - (-1)^2}{4} + 2.
\end{aligned}
$$

and

$$
\begin{aligned}
|S_{1,0}^r| &= \frac{3^{r-1} + (-1)^{r-1}}{2} + |S_{1,0}^{r-1}| \\
&= \frac{3^{r-1} + (-1)^{r-1}}{2} + ... + \frac{3^2 + (-1)^2}{2} + |S_{1,0}^2| \\
&= \frac{3^{r-1} + ... + 3^2}{2} + \frac{(-1)^{r-1} + ... + (-1)^2}{2} + 2 \\
&= \frac{3^r - 3^2}{4} - \frac{(-1)^r - (-1)^2}{4} + 2.
\end{aligned}
$$

Appendix E: Calculation of $|T_{p,q}^{r,i}|$

It is easy to see that

$$
\begin{aligned}
|T_{p,q}^{r,0}| &= |U_{p\oplus 1,q}^{r-1}| \\
|T_{p,q}^{r,1}| &= |U_{p\oplus 1,q\oplus 1}^{r-1}| \\
|T_{p,q}^{r,2}| &= |S_{p,q\oplus 1}^{r-1}|,
\end{aligned}
$$

where

$$
U_{p,q}^r \equiv \left\{ (s_1, ..., s_r) \in \{0,1,2\}^{\times r} \,\middle|\, f(s_1, ..., s_r) = p \text{ and } g(s_1, ..., s_r) = q \right\}.
$$

Since

$$
\begin{aligned}
|T_{0,q}^{r,0}| &= |U_{1,q}^{r-1}| = \frac{1}{4}(3^{r-1} - (-1)^{r-1}) \\
|T_{0,q}^{r,1}| &= |U_{1,q\oplus 1}^{r-1}| = \frac{1}{4}(3^{r-1} - (-1)^{r-1}) \\
|T_{0,q}^{r,2}| &= |S_{0,q\oplus 1}^{r-1}| = \frac{1}{4}(3^{r-1} - 3^2 + (-1)^{r-1} - (-1)^2) + 2,
\end{aligned}
$$

we obtain $|T_{0,q}^{r,0}| = |T_{0,q}^{r,1}|$ and

$$
\begin{aligned}
|T_{0,q}^{r,0}| - |T_{0,q}^{r,2}| &= -\frac{(-1)^{r-1}}{2} + \frac{3^2}{4} + \frac{1}{4} - 2 \\
&= \frac{1 - (-1)^{r-1}}{2} \\
&= \begin{cases} 0 & (r = \text{odd}) \\ 1 & (r = \text{even}). \end{cases}
\end{aligned}
$$

Since

$$
\begin{aligned}
|T_{1,0}^{r,0}| &= |U_{0,0}^{r-1}| = \frac{1}{4}(3^{r-1} + 3(-1)^{r-1}) \\
|T_{1,0}^{r,1}| &= |U_{0,1}^{r-1}| = \frac{1}{4}(3^{r-1} - (-1)^{r-1}) \\
|T_{1,0}^{r,2}| &= |S_{1,1}^{r-1}| = \frac{3^{r-1} - 3^2}{4} - \frac{(-1)^{r-1} - (-1)^2}{4} + 2,
\end{aligned}
$$

we obtain

$$
|T_{1,0}^{r,0}| - |T_{1,0}^{r,1}| = (-1)^{r-1},
$$

$$
\begin{aligned}
|T_{1,0}^{r,0}| - |T_{1,0}^{r,2}| &= \frac{3^2}{4} + (-1)^{r-1} - \frac{1}{4} - 2 \\
&= (-1)^{r-1},
\end{aligned}
$$

and

$$|T_{1,0}^{r,1}| - |T_{1,0}^{r,2}| = \frac{3^2}{4} - \frac{1}{4} - 2$$
$$= 0.$$

Since

$$|T_{1,1}^{r,0}| = |U_{0,1}^{r-1}| = \frac{1}{4}(3^{r-1} - (-1)^{r-1})$$
$$|T_{1,1}^{r,1}| = |U_{0,0}^{r-1}| = \frac{1}{4}(3^{r-1} + 3(-1)^{r-1})$$
$$|T_{1,1}^{r,2}| = |S_{1,0}^{r-1}| = \frac{3^{r-1} - 3^2}{4} - \frac{(-1)^{r-1} - (-1)^2}{4} + 2,$$

we obtain

$$|T_{1,1}^{r,0}| - |T_{1,1}^{r,1}| = -(-1)^{r-1},$$

$$|T_{1,1}^{r,0}| - |T_{1,1}^{r,2}| = \frac{3^2}{4} - \frac{1}{4} - 2$$
$$= 0,$$

and

$$|T_{1,1}^{r,1}| - |T_{1,1}^{r,2}| = \frac{3^2}{4} + (-1)^{r-1} - \frac{1}{4} - 2$$
$$= (-1)^{r-1}.$$

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[35] This result is reasonable since we physically do a non-linear (or non-TP) operation. If we consider this fact, it is surprising that linear CPTP operations are implemented by doing non-linear (or non-TP) physical operations on several resource states such as the cluster state when physical operations are perfect!